Spectral properties of fluctuating electromagnetic fields in a plane cavity: Implication for nanoscale physics

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Spectral power densities of fluctuating electromagnetic fields and their spatial derivatives of all orders in any point of a transparent plane gap between two media described by different complex permittivities and by different temperatures were derived on a basis of generalized Kirchhoff's law. Electromagnetic losses into the two absorbing media induced by a field of a point dipole or of point multipolelike origins situated in any place of interest at the transparent gap were determined. The corresponding electrodynamical regular Green problem for a point dipole and for point multipoles of any orders constituted by the point dipole was solved. We demonstrate ways to obtain different asymptotic cases following from our general solution including the problem for a half space, Planck's formula for black body radiation, the van der Waals forces for solids kept at different temperatures, and contributions from propagating and evanescent waves. Expressions for electromagnetic loss of a point multipole of any order in selected geometry of the problem were derived and, as an important limiting case related to problems of near field microscopy, when the multipole is situated over a half space.

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I. INTRODUCTION

A variety of important phenomena exists in physics of systems with a confined geometry of space. For instance, the Casimir effect [1], cavity induced processes [2,3], evanescent waves in nanoscale physics [4], and thermal and quantum electromagnetic fluctuations [5,6]. In practically all of the above mentioned phenomena a spectral composition of electromagnetic fields (EMF) plays a major role. In order to solve some related problems, we have to find spectral power densities (SPD) of electromagnetic fields and their spacial derivatives in a cavity of various geometry or in a bounded space.

The knowledge of spatial derivatives permits us to find any forces acting on multipole systems in a cavity or near a surface. A potential energy of a system in a field $\vec{E}$ characterized by a potential ϕ of another system may be represented as a series,

$$U = e\phi - p_i E_i + q_{ik}\frac{\partial E_i}{\partial x_k} + \cdots \tag{1}$$

where e is the charge, $\vec{p}$ the dipole moment, and q_{ik} the quadrupole moment of a system of interest, etc. If we know the multipole polarizabilities of a system we may calculate a force acting on the uncharged system in a stochastic field:

$$f_i = -\frac{\partial}{\partial x_i}\left\{ -\alpha_{ik}^{(p)}\langle E_i E_k\rangle + \alpha_{ikj}^{(q)}\left\langle E_j \frac{\partial E_i}{\partial x_k}\right\rangle + \cdots \right\}. \tag{2}$$

The correlation functions $\langle E_i E_k\rangle$, $\langle E_j \partial E_i / \partial x_k\rangle$ and others may depend on geometrical parameters of a bounded space.

Besides conservative forces like (2), a moving system may be subjected to dissipative forces. In a free space, friction may occur via interaction with the black body radiative field or with a zero-point fluctuating field. Properties of fluctuating electromagnetic fields in this case are known quite well, but in reality, a particle is moving near a wall or in channels of various geometrical forms, in other words, in a bounded space. In this case the dissipative and conservative interactions are occurring via both radiative and evanescent fields whose properties are determined by geometrical peculiarities of the problem and by electromagnetic properties of a matter-constituted space. For example, the dissipative interaction of a mobile particle with a half space was considered in a relativistic approach on the basis of fluctuating electrodynamics by Dorofeyev *et al.* [7]. In order to study similar problems with other geometrical conditions we need to investigate properties of fluctuating fields in geometries of interest.

Other intriguing problems in science are connected with the influence of surrounding matter, including geometrical conditions ("background"), on the probability of a quantum transition from state −1 to state 2. We may consider this background, for instance, as a stochastic action of other systems, of random force fields of any nature, and other fluctuating action. The influence of such stochastic action and interaction of the quantum system with surrounding systems may be described by the fluctuating part of Hamiltonian $U(t,z)$ as a random function of time and a regular (or random!) function of geometrical properties, which is described by the parameter z. It may be, for example, a value of a gap between two slabs, etc. In this case the complete Hamiltonian of a quantum system plus fluctuating background has the form

$$\hat{H} = \hat{H}_0 + \hat{U}(t,z), \tag{3}$$

where $\hat{H}_0$ is the Hamiltonian of an isolated quantum system or a quantum system in a free space.

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Using the well-known formula from [8] for a probability amplitude $C_{2\leftarrow 1}$ of a transition during the time $(t-t_0)$, we obtain the probability of a quantum transition averaged out over an ensemble of realization of interaction with the stochastic "background" $P_{2\leftarrow 1}=|C_{2\leftarrow 1}|^2$ which is proportional to the value $\langle U_{21}(t')U_{21}(t'')\rangle$, where $U_{21}(t)$ is the matrix element of the $(2\leftarrow 1)$ transition.

In the case of a weak electromagnetic field in the dipole approach $U\simeq -\vec{p}\vec{E}$, induced transitions in a stochastic field are proportional to a correlation tensor of the field $\langle E_i(t')E_k(t'')\rangle$. The velocity of a transition $W=P(t)/(t-t_0)$ in steady state stochastic fields is proportional to a SPD at the frequency of the transition

$$W_{2\leftarrow 1}=2\pi\hbar^{-2}\sum_{ik} p_i p_k^* G_{ik}(\omega_{21}), \tag{4}$$

where $G_{ik}(\omega)$ is the Fourier transform of the correlation tensor $G_{ik}(t'-t'')=\langle E_i(t')E_k(t'')\rangle$ and the asterisk denotes a complex conjugation. $G_{ik}(\omega,z)$ may function on a geometrical parameter z of a problem.

Another related example is connected with the problem of energy level shifts of a quantum system under the action of a stochastic EMF. In accordance with second order perturbation theory, the shift of nth level equals

$$\delta\mathbf{E}_n=-\sum_m{}' \frac{(p_i)_{nm}(p_k)_{mn}}{\mathbf{E}_n-\mathbf{E}_m}\langle E_iE_k\rangle, \tag{5}$$

where the prime means $m\neq n$.

A lot of important problems in nanophysics are connected with processes originating from near field interaction of bodies. For instance, in scanning probe microscopy it is often necessary to understand reversible or irreversible changing at the surface under various kinds of action from the side of a probe. In the case of a near field optical microscope, it may be photoexitation and heating followed by physical-chemical phenomena at the surface. One problem here is to calculate the energy release rate into a sample when the radiating system may be represented as a multipole system. Results of our paper may find useful applications in this area.

The goal of this paper is to calculate spectral power densities of fluctuating electromagnetic fields and their spatial derivatives in a plane gap consisting of two half spaces with different complex permittivity and kept at different temperatures. We based our calculation on the generalized Kirchhoff's law [5,6]. In accordance with this law we solve the corresponding Green problem and related problems for multipolelike origins for selected geometry of our problem. Limiting cases of our general solution we consider various asymptotic results. In particular, we consider the following. (a) We consider two cases for propagating and evanescent waves expressing the solution via ordinary surface Fresnel coefficients. (b) We demonstrate a transition to the case for a half space and compare with the known solution for SPD of components of a fluctuating EMF in this case. (c) We show how to obtain a general expression for the van der Waals force for the general case of different temperatures of interacted bodies. (d) We find a rate of energy liberation into a half space induced by a radiating system which may be represented by infinite multipole series.

II. PROBLEM STATEMENT

Here we use the reciprocity theorem to find components of a fluctuating field and its spatial derivatives with the help of a point dipolelike or multipolelike origin. At first we consider two antiparallel point origins situated near each other at the spatial points $\vec{r}$ and $\vec{r}+\Delta\vec{s}$. For example, electric components of a fluctuating field along the point dipoles are connected with fields of the dipoles by the formulas

$$E_\alpha(\vec{r})=\int(\vec{\mathcal{E}}_d\cdot\vec{j}_e-\vec{\mathcal{H}}_d\cdot\vec{j}_m)d\vec{r}, \tag{6}$$

$$-E_\alpha(\vec{r}+\Delta\vec{s})=\int(\vec{\mathcal{E}}_{d'}\cdot\vec{j}_e-\vec{\mathcal{H}}_{d'}\cdot\vec{j}_m)d\vec{r}, \tag{7}$$

where $\vec{\mathcal{E}}_d,\vec{\mathcal{H}}_d$ and $\vec{\mathcal{E}}_{d'},\vec{\mathcal{H}}_{d'}$ are the fields of the two antiparallel dipoles, $\vec{p}=-\vec{p}\,'$, and $\vec{j}_e$ and $\vec{j}_m$ are the densities of the external electric and magnetic currents that are creating the fluctuating fields. α may be x, y, or z. Summing up these two equations and considering the limit $|\Delta\vec{s}|\rightarrow 0$ we have a derivative of the component E_α of a fluctuating field along the direction $\vec{s}$

$$\frac{\partial E_\alpha}{\partial_s}=\int(\vec{\mathcal{E}}_1^q\cdot\vec{j}_e-\vec{\mathcal{H}}_1^q\cdot\vec{j}_m)d\vec{r}, \tag{8}$$

where

$$\vec{\mathcal{E}}_1^q=\lim_{|\Delta\vec{s}|\rightarrow 0}\frac{\vec{\mathcal{E}}_d+\vec{\mathcal{E}}_{d'}}{|\Delta\vec{s}|} \quad\text{and}\quad \vec{\mathcal{H}}_1^q=\lim_{|\Delta\vec{s}|\rightarrow 0}\frac{\vec{\mathcal{H}}_d+\vec{\mathcal{H}}_{d'}}{|\Delta\vec{s}|},$$

the strengths of a field of quadrupolelike point origin.

Thus, different point quadrupoles determine different first spatial derivatives. For example, a quadrupole of electric origin determines spatial derivatives of the electric field

$$q_{xz}\rightarrow\frac{\partial E_x}{\partial z},\quad q_{yz}\rightarrow\frac{\partial E_y}{\partial z},\quad q_{zz}\rightarrow\frac{\partial E_z}{\partial z}. \tag{9}$$

and, correspondingly, a magnetic quadrupole determines spatial derivatives of the magnetic field. In the same way, it is possible to find derivations of any order

$$\frac{\partial^n E_\alpha}{\partial_{s1}\cdots\partial s_n}=\int(\vec{\mathcal{E}}_n^q\cdot\vec{j}_e-\vec{\mathcal{H}}_n^q\cdot\vec{j}_m)d\vec{r}, \tag{10}$$

where

$$\vec{\mathcal{E}}_n^q=\lim_{|\Delta\vec{s}_n|\rightarrow 0}\frac{\vec{\mathcal{E}}_{n-1}^q+\vec{\mathcal{E}}_{n-1}^{q\,\prime}}{|\Delta\vec{s}_n|}$$

and

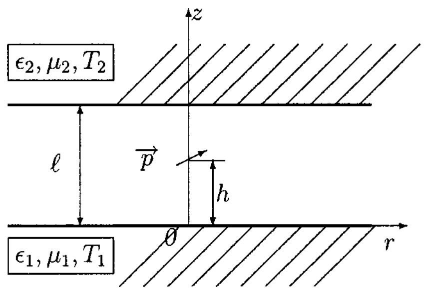

FIG. 1. Representation of two absorbing homogeneous isotropic media.

$$\vec{\mathcal{H}}_n^q = \lim_{|\Delta \vec{s}_n| \to 0} \frac{\vec{\mathcal{H}}_{n-1}^q + \vec{\mathcal{H}}_{n-1}^{q'}}{|\Delta \vec{s}_n|},$$

the strengths of a field of the multipolelike point origin.

In this way we are creating special kinds of multipoles with the help of a point dipole located in the selected directions. In order to find the component of a fluctuating field or any spatial derivative we need to solve a corresponding problem for a pointlike dipole or multipole origin.

Multiplication of any component or derivative (A_α) from (6,10) to a complex conjugated (B_β^*) from (6,10) followed by application of the electrodynamical fluctuation-dissipation theorem (FDT) [5,6] gives the generalized Kirchhoff's law,

$$\langle A_\alpha(\omega, \vec{r}_1) B_\beta^*(\omega, \vec{r}_2) \rangle = \frac{2}{\pi} \Theta(\omega, T) Q_{\mathrm{AB}*}(\vec{r}_1, \vec{r}_2), \quad (11)$$

where $Q_{\mathrm{AB}*}$ is the mixed electromagnetic loss of unit multipoles located in the spatial points $\vec{r}_1$ and $\vec{r}_2$, correspondingly, $\Theta(\omega, T) = (\hbar\omega/2)\coth(\hbar\omega/2k_B T)$, k_B is the Boltzmann constant, and the brackets represent averaging over an ensemble. Thus, in order to obtain $\langle |E_\alpha|^2 \rangle$, $\langle E_\alpha \partial E_\alpha^* / \partial \beta \rangle$ or $\langle |\partial E_\alpha / \partial \beta|^2 \rangle$, we need to calculate, correspondingly, the electromagnetic losses of a unit dipole, mixed losses of a unit dipole–quadrupole system, and the losses of unit quadrupole as situated at the points of interest. The same statement obviously concerns any derivatives and multipoles.

We will seek the SPD of fields between two absorbing homogeneous isotropic media with different temperatures (Fig. 1). The half space $z \leqslant 0$ is filled with a material characterized by complex constants ϵ_1, μ_1, the half space $z \geqslant l$ is filled with a medium with the constants ϵ_2, μ_2, while the gap between them is a vacuum or filled with a nonabsorbing medium with the real dielectric constants $\dot{\varepsilon}, \mu$. We assume that two subsystems of extraneous random sources of fluctuating fields are located in two thermostats with temperatures T_1 and T_2 maintained constant. The system consists of equilibrium subsystems, justifying the application of FDT. The relaxation time of the overall system is possibly to be considered much larger than the relaxation times of two subsystems.

In our case using the reciprocity theorem and the principle of superposition of fields, we make the required square combinations from the components of the fluctuation field. Next, after averaging over the equilibrium ensembles of random currents, we apply the electrodynamic FDT to obtain mean square characteristics of the fluctuating field, expressed via the thermal losses of regular fields in either medium, induced by point dipole placed in the gap at some distance h from the lower medium and oriented in an appropriate fashion. As a result, multiplying the losses Q_1 in the first medium by Θ_1, and in the second medium (Q_2) by Θ_2, we have, for example, for z components of a fluctuating EMF, another form of the generalized Kirchhoff's law

$$\langle |E_i(\vec{r})|^2 \rangle = \frac{2}{\pi} \{ \Theta_1 Q_1(p_i^e; \vec{r}) + \Theta_2 Q_2(p_i^e, \vec{r}) \},$$

$$\langle |H_i(\vec{r})|^2 \rangle = \frac{2}{\pi} \{ \Theta_1 Q_1(p_i^m; \vec{r}) + \Theta_2 Q_2(p_i^m, \vec{r}) \}, \quad (12)$$

where notations p_i^e and p_i^m indicate the requirement to find the losses of regular fields induced by electric and magnetic point dipoles, respectively, $i = x, y, z$. It directly follows from the above discussions that the related expressions for any spatial derivative of fluctuating EMF using multipoles composed of a point dipole can be obtained.

Further, we separate the electromagnetic losses from the even and odd multipoles, for convenience. We introduce the following notations for even derivatives of field components:

$$\left\langle \left| \frac{\partial^m E_z}{\partial z^m}(\vec{r}, \omega) \right|^2 \right\rangle \equiv \langle |E_z^{(m)}(\vec{r}, \omega)|^2 \rangle \equiv \langle |E_z^{(m)}|^2 \rangle,$$

$$\left\langle \left| \frac{\partial^m E_{x,y}}{\partial z^m}(\vec{r}, \omega) \right|^2 \right\rangle \equiv \langle |E_{x,y}^{(m)}(\vec{r}, \omega)|^2 \rangle \equiv \langle |E_{x,y}^{(m)}|^2 \rangle, \quad (13)$$

where $m = 2k,\ k = 0, 1, \ldots$.

A very important case, $m = 0$, corresponds to components of EMF, namely,

$$\langle |E_z^{(0)}|^2 \rangle \equiv \langle |E_z|^2 \rangle, \quad \langle |E_{x,y}^{(0)}|^2 \rangle \equiv \langle |E_{x,y}|^2 \rangle. \quad (14)$$

For odd derivatives we have the same notations, but with another letter, $n = 2k + 1$, $k = 0, 1, \ldots$. For mixed derivatives we have, for example, for the z component of a field,

$$\left\langle \frac{\partial^m E_z}{\partial z^m}(\vec{r}, \omega) \frac{\partial^n E_z^*}{\partial z^n}(\vec{r}, \omega) \right\rangle \equiv \langle E_z^{(m)}(\vec{r}, \omega) E_z^{(n)*}(\vec{r}, \omega) \rangle \equiv \langle E_z^{(m)} E_z^{(n)*} \rangle. \quad (15)$$

For components of a magnetic field we have similar notations. Everywhere in the paper we will use notations for a fluctuating field as $\vec{E},\vec{H}$ differing from the notations for a regular field $\vec{\mathcal{E}},\vec{\mathcal{H}}$, which are generated by pointlike multipole origins.

Then, for convenience we introduce mixed losses for even and odd multipole origins and seek any losses as an integral on the pointing vector all along, for example, surface $z=0$ by an integration in a cylindrical coordinates system

$$Q_{m,n}^{e,o}=-\frac{c}{16\pi}\int_0^\infty r\,dr\int_0^{2\pi}d\varphi\{[(\vec{\mathcal{E}}_m^e+\vec{\mathcal{E}}_n^o)\times(\vec{\mathcal{H}}_m^{e*}+\vec{\mathcal{H}}_n^{o*})]_z + \text{c.c.}\}_{z=0}, \tag{16}$$

where $\vec{\mathcal{E}}_{m,n}^{e,o}$ and $\vec{\mathcal{H}}_{m,n}^{e,o}$ and the fields are created by even or odd multipole pointlike origins of m or n orders (c.c. means complex conjugation). So, from (16) we have any electromagnetic losses originating from pointlike multipoles

$$Q_{m,n}^{e,o}=Q_{mm}^{ee*}+Q_{m,n}^{e*o}+Q_{m,n}^{e,o*}+Q_{nn}^{oo*}, \tag{17}$$

where Q_{mm}^{ee*} are the losses of even m-order multipoles, $Q_{m,n}^{e*o}$ and $Q_{m,n}^{e,o*}$ the mixed even-odd multipole losses of m and n orders correspondingly, and Q_{nn}^{oo*} the n-order odd multipole losses. Obviously, similar to formula (16) but with an opposite sign is valid for loss calculations in a second half space.

III. SOLUTION TO THE PROBLEM

A. Multipole representations

As a first step we determine the Helmholtz equation for multipoles composed of point dipoles situated in directions of interest. As shown in Appendix A from the Maxwell system of equations and selected relationships of the Hertz vector with scalar and vector potentials, it follows the Helmholtz equation for a dipole origin is

$$\Delta\vec{Z}_d+k^2\vec{Z}_d=-4\pi\mu\vec{p}\,\delta(\vec{r}), \tag{18}$$

where $\vec{p}\,\delta(\vec{r})$ is the unit pointlike dipole origin of a field which is determined by the Hertz vector $\vec{Z}_d$. This is the ordinary Green problem. For another point dipole, $\vec{p}\,'$, at a distance $\vec{r}+\Delta\vec{s}_1$ from the first one, we have another equation:

$$\Delta\vec{Z}_{d'}+k^2\vec{Z}_{d'}=-4\pi\mu\vec{p}\,'\,\delta(\vec{r}+\Delta\vec{s}_1). \tag{19}$$

After summing up (18) with (19) and decomposing $\delta(\vec{r}+\Delta\vec{s}_1)\simeq\delta(\vec{r})+\Delta\vec{s}\cdot grad\,\delta(\vec{r})$ we have an equation for a quadrupole composed of two antiparallel pointlike dipoles $\vec{p}$ and $\vec{p}\,'=-\vec{p}$ and separated by a distance $|\Delta\vec{s}_1|$ along the direction $\vec{s}_1$,

$$\Delta\vec{Z}_1^q+k^2\vec{Z}_1^q=4\pi\mu\vec{p}\frac{\partial\delta(\vec{r})}{\partial s_1}, \tag{20}$$

where

$$\vec{Z}_1^q=\lim_{|\Delta\vec{s}_1|\to 0}\frac{\vec{Z}_d+\vec{Z}_{d'}}{|\Delta\vec{s}_1|}.$$

By a similar means it is possible to obtain the equation for any multipole origin,

$$\Delta\vec{Z}_n^q+k^2\vec{Z}_n^q=-(-1)^n 4\pi\mu\vec{p}\frac{\partial^n\delta(\vec{r})}{\partial s_1\cdots\partial s_n}, \tag{21}$$

where

$$\vec{Z}_n^q=\lim_{|\Delta\vec{s}_n|\to 0}\frac{\vec{Z}_{n-1}^q+\vec{Z}_{n-1}^{q\,'}}{|\Delta\vec{s}_n|}.$$

As usual, we will seek a general solution to Eq. (21) as a sum of a partial solution of (21) and a general solution of the corresponding homogeneous equation. In order to find a partial solution we use the Green function $\vec{Z}_d=\mu\vec{p}\,\exp(-ik|\vec{r}|)/|\vec{r}|$ of the problem (18). Using properties of generalized functions it is possible to find a corresponding solution of the problem (21):

$$\vec{Z}_n^q=(-1)^n\mu\vec{p}\int_{V'}\frac{\exp(-\iota k|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}\frac{\partial^n\delta(\vec{r}\,')}{\partial s_1'\cdots\partial s_n'}dV' =(-1)^n\mu\vec{p}\frac{\partial^n}{\partial s_1\cdots\partial s_n}\frac{\exp(-\iota k|\vec{r}|)}{|\vec{r}|}. \tag{22}$$

Finally, as follows from (22), in order to find a partial solution for any multipole we should take the corresponding derivatives with respect to the Green's function for free space.

B. Integral representation of multipole origins

To solve the problem (21) we need to find an integral representation of the corresponding partial solution. For a dipole we have the well-known Sommerfeld representation [14],

$$\frac{\exp(-\iota kR)}{R}=\int_0^\infty J_0(\lambda r)\frac{\exp(-q|z-h|)}{q}\lambda\,d\lambda, \tag{23}$$

where J_0 is the Bessel function of the zero order, $q=\sqrt{\lambda^2-k^2}$, $R=\sqrt{x^2+y^2+(z-h)^2}$, and $r=\sqrt{x^2+y^2}$ in accordance with Fig. 1.

Since we have homogeneous conditions along the x, y directions (see Fig. 1), we seek only z derivatives. For a quadrupole using the Sommerfeld representation (23) we have

$$\frac{\partial}{\partial z}\left(\frac{\exp(-\iota kR)}{R}\right)=\frac{\partial}{\partial z}\left\{\int_0^\infty J_0(\lambda r)q^{-1}\exp(-q|z-h|)\lambda\, d\lambda\right\}$$
$$=\begin{cases} -\int_0^\infty J_0(\lambda r)\exp[-q(z-h)]\lambda\, d\lambda & (z>h)\\ \int_0^\infty J_0(\lambda r)\exp[-q(h-z)]\lambda\, d\lambda & (z<h).\end{cases} \tag{24}$$

Using the function sgn(x)=1 ($x>0$), sgn(x)=0, ($x=0$), sgn(x)=−1 ($x<0$), and known relations for generalized functions $[\mathrm{sgn}(x)]^{(1)}=2\delta(x)$ and $f(x)\delta(x)=f(0)\delta(x)$ we can write the result in the compact form

$$\frac{\partial^n}{\partial z^n}\frac{\exp(-\iota kR)}{R}=[\mathrm{sgn}(h-z)]^n\int_0^\infty q^{n-1}J_0(\lambda r)\times\exp(-q|z-h|)\lambda\, d\lambda+\sum_{j=0}^{N}[\mathrm{sgn}(h-z)]^{(n-2j-1)}\times\int_0^\infty q^{2j}J_0(\lambda r)\lambda\, d\lambda,\quad n=0,1,\ldots, \tag{25}$$

where $N=(n-2)/2$ for $n=2k$ and $N=(n-3)/2$ for $n=2k+1$, $k=0,1,2,\ldots$. At negative N the sum in (25) is zero.

Thus, we obtain the integral representation of the partial solution for a multipole origin of any order $n=0,1,\ldots$, namely, $\vec{Z}_n^q=(-1)^n\mu\vec{p}Z_n^q$, where

$$Z_n^q=\left\{[\mathrm{sgn}(h-z)]^n\int_0^\infty q^{n-1}J_0(\lambda r)\exp(-q|z-h|)\lambda\, d\lambda+\sum_{j=0}^{N}[\mathrm{sgn}(h-z)]^{(n-2j-1)}\int_0^\infty q^{2j}J_0(\lambda r)\lambda\, d\lambda\right\}, \tag{26}$$

where the case $n=0$ corresponds to the classical Sommerfeld integral representation for a dipole origin.

C. Spectral characteristics of a fluctuating field

As follows from (11) and (12), in order to obtain some spectral characteristic, we ought to determine the electromagnetic loss of a point multipole into both half spaces. For this purpose (see Appendix B) we found the electromagnetic field of a multipole as situated in any point of the transparent gap. Then, in accordance with (16) we had to seek the losses (see Appendix C).

Using calculated losses from Appendix C with the corresponding coefficients from Appendix B we find the spectral power densities of a field and spatial derivatives, as established by the generalized Kirchhoff's law (12).

For even derivatives of z components of an electric field

$$\langle|E_z^{(m)}|^2\rangle=\frac{2\Theta_1}{\pi\omega\epsilon}\mathrm{Re}\int_0^\infty\frac{\lambda^3\, d\lambda}{\iota q^*}A_m^e+\frac{2\Theta_2}{\pi\omega\epsilon}\mathrm{Re}\int_0^\infty\frac{\lambda^3\, d\lambda}{\iota q^*}B_m^e, \tag{27}$$

where

$$A_m^e=|q^m|^2\frac{\beta_1}{\tilde{\alpha}_1}\frac{\left|\cosh(ql-qh)+\frac{\beta_2}{\tilde{\alpha}_2}\sinh(ql-qh)\right|^2}{|\tilde{D}|^2},$$

$$B_m^e=|q^m|^2\frac{\beta_2}{\tilde{\alpha}_2}\frac{\left|\cosh(qh)+\frac{\beta_1}{\tilde{\alpha}_1}\sinh(qh)\right|^2}{|\tilde{D}|^2},$$

$$m=2k,\quad k=0,1,\ldots. \tag{28}$$

For odd derivatives of z components of an electric field we have the similar formula (27), but with odd coefficients

$$A_n^0=|q^n|^2\frac{\beta_1}{\tilde{\alpha}_1}\frac{\left|\sinh(ql-qh)+\frac{\beta_2}{\tilde{\alpha}_2}\cosh(ql-qh)\right|^2}{|\tilde{D}|^2},$$

$$B_n^0=|q^n|^2\frac{\beta_2}{\tilde{\alpha}_2}\frac{\left|\sinh(qh)+\frac{\beta_1}{\tilde{\alpha}_1}\cosh(qh)\right|^2}{|\tilde{D}|^2},$$

$$n=2k+1,\quad k=0,1,\ldots. \tag{29}$$

For the even derivatives of x, y components of an electric field we found

$$\langle|E_{x,y}^{(m)}|^2\rangle=\frac{\Theta_1}{\pi\omega\epsilon}\mathrm{Re}\int_0^\infty\frac{\lambda\, d\lambda}{\iota q}\mathbf{C}_m^e+\frac{\Theta_2}{\pi\omega\epsilon}\mathrm{Re}\int_0^\infty\frac{\lambda\, d\lambda}{\iota q}\mathbf{D}_m^e, \tag{30}$$

where

$$\mathbf{C}_m^e = q^2 \frac{\beta_1}{\tilde{\alpha}_1} \tilde{C}_m^e - k^2 \frac{\beta_1^*}{\alpha_1^*} C_m^e$$

and

$$\mathbf{D}_m^e = q^2 \frac{\beta_2}{\tilde{\alpha}_2} \tilde{D}_m^e - k^2 \frac{\beta_2^*}{\alpha_2^*} D_m^e, \tag{31}$$

with coefficients (D1).

For the odd derivatives of the *x, y* components we have expression (30), but with odd terms

$$\mathbf{C}_n^o = q^2 \frac{\beta_1}{\tilde{\alpha}_1} \tilde{C}_n^o - k^2 \frac{\beta_1^*}{\alpha_1^*} C_n^o$$

and

$$\mathbf{D}_n^o = q^2 \frac{\beta_2}{\tilde{\alpha}_2} \tilde{D}_n^o - k^2 \frac{\beta_2^*}{\alpha_2^*} D_n^o, \tag{32}$$

and corresponding coefficients (D2). For mixed odd-even derivatives (o^*e) of the *z* components of a field we have

$$\langle E_z^{(m)} E_z^{(n)*} \rangle = \frac{2\Theta_1}{\pi\omega\epsilon} \mathrm{Re} \int_0^\infty \frac{\lambda^3\, d\lambda}{\iota q^*} \frac{\beta_1}{\tilde{\alpha}_1} A_{mn}^{o^*e} + \frac{2\Theta_2}{\pi\omega\epsilon} \mathrm{Re} \int_0^\infty \frac{\lambda^3\, d\lambda}{\iota q^*} \frac{\beta_2}{\tilde{\alpha}_2} B_{mn}^{o^*e}, \tag{33}$$

with expressions (D3).

For mixed even-odd derivatives (e^*o) of the *z* components we have the same formula (33), but with complex conjugated coefficients $A_{mn}^{e^*o} = (A_{mn}^{o^*e})^*$ and $B_{mn}^{e^*o} = (B_{mn}^{o^*e})^*$.

For mixed odd-even derivatives (o^*e) of the *x,y* components of a field we have

$$\langle E_{x,y}^{(m)} E_{x,y}^{(n)*} \rangle = \frac{\Theta_1}{\pi\omega\epsilon} \mathrm{Re} \int_0^\infty \frac{\lambda\, d\lambda}{\iota q} \mathbf{C}_{mn}^{o^*e} + \frac{\Theta_2}{\pi\omega\epsilon} \mathrm{Re} \int_0^\infty \frac{\lambda\, d\lambda}{\iota q} \mathbf{D}_{mn}^{o^*e}, \tag{34}$$

where

$$\mathbf{C}_{mn}^{o^*e} = k^2 \frac{\beta_1^*}{\alpha_1^*} C_{mn}^{o^*e} - q^2 \frac{\beta_1}{\tilde{\alpha}_1} \tilde{C}_{mn}^{o^*e}$$

and

$$\mathbf{D}_{mn}^{o^*e} = q^2 \frac{\beta_2}{\tilde{\alpha}_2} \tilde{D}_{mn}^{o^*e} - k^2 \frac{\beta_2^*}{\alpha_2^*} D_{mn}^{o^*e}, \tag{35}$$

and coefficients (D4).

We obtain corresponding formulas for mixed even-odd derivatives (e^*o) of *x,y* components of a field from (34), replacing $\mathbf{C}_{mn}^{e^*o}$ and $\mathbf{D}_{mn}^{e^*o}$ with the coefficients complex conjugated to (D4).

IV. DISCUSSION

A. Spectral power densities for propagating and evanescent waves in a plane cavity

Naturally, we must obtain any classical consequences from our general solution. We demonstrate corresponding results in a very important case for SPD components of fluctuating EMP. It directly follows from (27) and (28) for the case $m=0$. For convenience we introduce the distance $d=l-h$ between a second half space and a point of interest in the gap. After some obvious transformations obtained by eliminating terms $\exp(ql-qh)$ and $\exp(qh)$ from the numerators of A_m^e, B_m^e simultaneously with the term $\exp(ql)$ from denominators of A_m^e, B_m^e, we have for the *z* component of a field

$$\langle |E_z|^2 \rangle = \frac{\iota\Theta_1}{\pi c} \sqrt{\frac{\mu}{\epsilon}} \int_0^\infty \exp[-(q+q^*)h] \left\{ \frac{\lambda^2}{k^2} S_{1\epsilon} \mathbf{I}_{1\epsilon}^+ \right\} \lambda\, d\lambda + \frac{\iota\Theta_2}{\pi c} \sqrt{\frac{\mu}{\epsilon}} \int_0^\infty \exp[-(q+q^*)\, d] \times \left\{ \frac{\lambda^2}{k^2} S_{2\epsilon} \mathbf{I}_{2\epsilon}^+ \right\} \lambda\, d\lambda, \tag{36}$$

and for the *x, y* components

$$\langle |E_{x,y}|^2 \rangle = \frac{\iota\Theta_1}{2\pi c} \sqrt{\frac{\mu}{\epsilon}} \int_0^\infty \exp[-(q+q^*)h] \left\{ \frac{|q|^2}{k^2} S_{1\epsilon} \mathbf{I}_{1\epsilon}^- + S_{1\mu} \mathbf{I}_{1\mu}^+ \right\} \lambda\, d\lambda + \frac{\iota\Theta_2}{2\pi c} \sqrt{\frac{\mu}{\epsilon}} \int_0^\infty \exp[-(q+q^*)d] \left\{ \frac{|q|^2}{k^2} S_{2\epsilon} \mathbf{I}_{2\epsilon}^- + S_{2\mu} \mathbf{I}_{2\mu}^+ \right\} \lambda\, d\lambda, \tag{37}$$

where

$$S_{i\epsilon} = \frac{k(q_i^*/\epsilon_i^* - q_i/\epsilon_i)}{\epsilon |q/\epsilon + q_i/\epsilon_i|^2}, \quad S_{i\mu} = \frac{k(q_i^*/\mu_i^* - q_i/\mu_i)}{\mu |q/\mu + q_i/\mu_i|^2} \quad (i=1,2), \tag{38}$$

and

$$\mathbf{I}_{1\epsilon}^\pm = \left| \frac{1 \pm \mathbf{r}_2^p \exp[-2q(l-h)]}{1 - \mathbf{r}_1^p \mathbf{r}_2^p \exp(-2ql)} \right|^2,$$

$$\mathbf{I}_{2\epsilon}^\pm = \left| \frac{1 \pm \mathbf{r}_1^p \exp[-2q(l-d)]}{1 - \mathbf{r}_1^p \mathbf{r}_2^p \exp(-2ql)} \right|^2, \tag{39}$$

$$\mathbf{I}_{1\mu}^\pm = \left| \frac{1 \pm \mathbf{r}_2^s \exp[-2q(l-h)]}{1 - \mathbf{r}_1^s \mathbf{r}_2^s \exp(-2ql)} \right|^2,$$

$$\mathbf{I}_{2\mu}^\pm = \left| \frac{1 \pm \mathbf{r}_1^s \exp[-2q(l-d)]}{1 - \mathbf{r}_1^s \mathbf{r}_2^s \exp(-2ql)} \right|^2, \tag{40}$$

where

$$\mathbf{r}_i^p=\frac{(1-\beta_i/\tilde{\alpha}_i)}{(1+\beta_i/\tilde{\alpha}_i)}$$
$$=\frac{(\epsilon_1\sqrt{\omega^2\epsilon\mu/c^2-\lambda^2}-\epsilon\sqrt{\omega^2\epsilon_i\mu_i/c^2-\lambda^2})}{(\epsilon_i\sqrt{\omega^2\epsilon\mu/c^2-\lambda^2}+\epsilon\sqrt{\omega^2\epsilon_i\mu_i/c^2-\lambda^2})}\quad (i=1,2), \tag{41}$$

$$\mathbf{r}_i^s=\frac{(1-\beta_i/\alpha_i)}{(1+\beta_i/\alpha_i)}$$
$$=\frac{(\mu_i\sqrt{\omega^2\epsilon\mu/c^2-\lambda^2}-\mu\sqrt{\omega^2\epsilon_i\mu_i/c^2-\lambda^2})}{(\mu_i\sqrt{\omega^2\epsilon\mu/c^2-\lambda^2}+\mu\sqrt{\omega^2\epsilon_i\mu_i/c^2-\lambda^2})}\quad (i=1,2) \tag{42}$$

are the ordinary surface reflection Fresnel coefficients for p and s polarized waves, respectively.

Obviously, the terms $\mathbf{I}_{1\epsilon}^{\pm},\mathbf{I}_{2\epsilon}^{\pm}$ and $\mathbf{I}_{1\mu}^{\pm},\mathbf{I}_{2\mu}^{\pm}$ describe interference processes in the plane gap, similar to corresponding formulas of the Fabry-Perot system. Rearranging S_ϵ and S_μ, $\mathbf{I}_{1\epsilon}^{\pm},\mathbf{I}_{2\epsilon}^{\pm}$ and $\mathbf{I}_{1\mu}^{\pm},\mathbf{I}_{2\mu}^{\pm}$ in (36) and (37) gives an expression for the SPD of magnetic components, $\langle|H_z|^2\rangle$ and $\langle|H_{x,y}|^2\rangle$.

After that it is easy to obtain a formula for the spectral density of energy of the electric, $U_{e\omega}=\epsilon\langle|E|^2\rangle/4\pi$, and magnetic, $U_{m\omega}=\mu\langle|H|^2\rangle/4\pi$, parts of fluctuating EMF and the total spectral density of energy $U_\omega=U_{e\omega}+U_{m\omega}$ over positive frequencies

$$\begin{aligned}U_\omega=&\frac{\iota\Theta_1\sqrt{\epsilon\mu}}{4\pi^2c}\int_0^\infty \exp[-(q+q^*)h]\left\{\left(\frac{|q|^2\mathbf{I}_{1\epsilon}^-+\lambda^2\mathbf{I}_{1\epsilon}^+}{k^2}\right.\right.\\&\left.+\mathbf{I}_{1\epsilon}^+\right)S_{1\epsilon}+\left(\frac{|q|^2\mathbf{I}_{1\mu}^-+\lambda^2\mathbf{I}_{1\mu}^+}{k^2}+\mathbf{I}_{1\mu}^+\right)S_{1\mu}\Bigg\}\lambda\,d\lambda\\&+\frac{\iota\Theta_2\sqrt{\epsilon\mu}}{4\pi^2c}\int_0^\infty \exp[-(q+q^*)d]\left\{\left(\frac{|q|^2\mathbf{I}_{2\epsilon}^-+\lambda^2\mathbf{I}_{2\epsilon}^+}{k^2}\right.\right.\\&\left.+\mathbf{I}_{2\epsilon}^+\right)S_{2\epsilon}+\left(\frac{|q|^2\mathbf{I}_{2\mu}^-+\lambda^2\mathbf{I}_{2\mu}^+}{k^2}+\mathbf{I}_{2\mu}^+\right)S_{2\mu}\Bigg\}\lambda\,d\lambda.\end{aligned} \tag{43}$$

As was shown by different authors, the quantity $q=\sqrt{\lambda^2-k^2}$ determines propagating and evanescent waves. Further, we study the simplest asymptotic $l\to\infty$ that can be found analytically. Of course, the general case for any value of the gap requires numerical solutions.

Large l physically means that l is much larger than all wavelengths determining a fluctuating EMF of a body. Then we may neglect all interference processes and put $\mathbf{I}_{1\epsilon}^{\pm}=\mathbf{I}_{2\epsilon}^{\pm}=\mathbf{I}_{1\mu}^{\pm}=\mathbf{I}_{2\mu}^{\pm}\simeq 1$ in (39) and (40) and in all corresponding formulas (36), (37), and (43). In this approximation we consider two essentially different waves.

1. Propagating waves and black body radiation

We consider an equilibrium case $\Theta_1=\Theta_2=\Theta$ in the limit $l\to\infty$. For propagating waves we have $\lambda<k$, when $q+q^*=0$ and $|q|^2=k^2-\lambda^2$. In this case the spectral density of energy U_ω is not dependent on a distant h or d in the gap and from (43) we have

$$U_\omega=\frac{\iota\Theta\sqrt{\epsilon\mu}}{2\pi^2c}\left\{\int_0^\infty(S_{1\epsilon}+S_{1\mu})\lambda\,d\lambda+\int_0^\infty(S_{2\epsilon}+S_{2\mu})\lambda\,d\lambda\right\}. \tag{44}$$

As in [5], we introduce the relative indexes of refraction $N_i=\sqrt{\epsilon_iu_i/\epsilon\mu}$, $i=1,2$, and the variable $x=\lambda/k=\sin\theta$, where θ is the angle of incidence of a propagating wave on absorbing matters. With these notations we rewrite (38) in the form

$$S_{j\epsilon}=-\frac{\iota(1-R_\parallel^j)}{2\sqrt{1-x^2}}\quad\text{and}\quad S_{j\mu}=-\frac{\iota(1-R_\perp^j)}{2\sqrt{1-x^2}}\quad (j=1,2), \tag{45}$$

where

$$\begin{aligned}R_\parallel^i=|\mathbf{r}_i^p|^2=&\left|\frac{\tilde{\alpha}_i\sqrt{1-x^2}-\sqrt{N_i^2-x^2}}{\tilde{\alpha}_i\sqrt{1-x^2}+\sqrt{N_i^2-x^2}}\right|^2\\=&\left|\frac{\tilde{\alpha}_i\cos\theta-\sqrt{N_i^2-\sin^2\theta}}{\tilde{\alpha}_i\cos\theta+\sqrt{N_i^2-\sin^2\theta}}\right|,\\R_\perp^i=|\mathbf{r}_i^s|^2=&\left|\frac{\alpha_i\sqrt{1-x^2}-\sqrt{N_i^2-x^2}}{\alpha_i\sqrt{1-x^2}+\sqrt{N_i^2-x^2}}\right|\\=&\left|\frac{\alpha_i\cos\theta-\sqrt{N_i^2-\sin^2\theta}}{\alpha_i\cos\theta+\sqrt{N_i^2-\sin^2\theta}}\right|^2,\end{aligned} \tag{46}$$

where $R_\parallel^i$ and $R_\perp^i$ are the energetic reflection coefficients from absorbing half spaces at the angle θ for waves with the electric vector in parallel and, correspondingly, perpendicular with respect to the plane of incidence and $i=1,2$, as usual. Because both polarizations of waves of stochastic EMF are absolutely equivalent, the total reflection coefficients are equal, $R_i=(R_\parallel^i+R_\perp^i)/2$. As a result we have for propagating waves

$$\begin{aligned}U_\omega=&\frac{\Theta\sqrt{\epsilon\mu}k^2}{2\pi^2c}\left\{\int_0^1(1-R_1)\frac{x\,dx}{\sqrt{1-x^2}}\right.\\&\left.+\int_0^1(1-R_2)\frac{x\,dx}{\sqrt{1-x^2}}\right\}.\end{aligned} \tag{47}$$

For the black body case we should put $R^i=0$, $(i=1,2)$ by definition. Finally, we have the well-known formula

$$U_\omega=\frac{\Theta k_0^2}{\pi^2c}n^3=U_{0\omega}n^3, \tag{48}$$

where $n=\sqrt{\epsilon\mu}$ is the index of refraction of the transparent media in the gap, and the spectral density of energy of black body radiation in a vacuum is

$$U_{0\omega}=\frac{\hbar\omega^3}{\pi^2c^3}\left(\frac{1}{2}+\frac{1}{\exp(\hbar\omega/k_BT)-1}\right). \tag{49}$$

2. Evanescent waves and related asymptotics

These waves are defined by the relation $\lambda>k$, when $q-q^*=0$ and $|q|^2=\lambda^2-k^2$. With the same approximation $l\to\infty$ after introducing of the variables $y=q/k=\sqrt{\lambda^2/k^2-1}$, we obtain from (43) the spectral density of energy for evanescent waves:

$$U_\omega^{ev}=\frac{\Theta_1k_0^2}{2\pi^2c}n^3\int_0^\infty \exp[-2khy](S_{1\epsilon}+S_{1\mu})(y^2+1)y\,dy+\frac{\Theta_2k_0^2}{2\pi^2c}n^3\int_0^\infty \exp[-2k\,dy](S_{2\epsilon}+S_{2\mu})(y^2+1)y\,dy. \tag{50}$$

For $\Theta_1=\Theta_2=\Theta$ using an estimation of the $S_{i\epsilon}$ where $S_{i\mu}$ as was done in [5,6] or in [8] for the case where relative refraction indexes of the half spaces are not very large, we found with an exactness up to the terms of order $1/(kh)^2$ and $1/(kd)^2$

$$U_\omega^{ev}\approx U_{0\omega}n^3\left\{\frac{(\alpha_1''/1+\tilde\alpha_1|^2+\alpha_1''/|1+\alpha_1|^2)}{[k(l-d)]^3}+\frac{(\tilde\alpha_2''/|1+\tilde\alpha_2|^2+\alpha_2''/|1+\alpha_2|^2)}{[k(l+h)]^3}\right\}. \tag{51}$$

It should be emphasized that (51) is valid for large gap values. To study an evanescent field structure in the arbitrary value of l we have to use the general formula (43).

B. Spectral power densities of fluctuating EMF for a half space

Another important related problem is the case of a half space when $l=\infty$, $\tilde\alpha_1=\tilde\alpha_2$, $\alpha_1=\alpha_2$, and $T_1=T_2$. In such a case we will consider any characteristics at a finite distance h. It is clear that the coefficients $B_{m,n}^{e,o}$, $\mathbf{D}_{m,n}^{e,o}$, B_{mn}^{o*e}, $\mathbf{D}_{mn}^{e*o}$, B_{mn}^{o*e} are equal to zero and we have SPD of a fluctuating EMF and its derivatives at the distance h over a half space described by complex permittivities ϵ_1 and μ_1. We demonstrate the result in the familiar form of SPD for components of fluctuating EMF over a half space. It follows directly from (27), (30), or (43) for the case $m=0$. After some obvious transformations we have

$$\langle|E_z|^2\rangle=\frac{2\Theta}{\pi\omega\epsilon}\mathrm{Re}\int_0^\infty\frac{\lambda^3\,d\lambda}{\iota q^*}\,\frac{\beta_1}{\tilde\alpha_1}\,\frac{\exp[-(q+q^*)h]}{|1+\beta_i/\tilde\alpha_1|^2}, \tag{52}$$

$$\langle|E_{x,y}|^2i\rangle=\frac{\Theta}{\pi\omega\epsilon}\mathrm{Re}\int_0^\infty\frac{\lambda\,d\lambda}{\iota q}\left\{q^2\frac{\beta_1}{\tilde\alpha_1}\,\frac{\exp[-(q+q^*)h]}{|1+\beta_1/\tilde\alpha_1|^2}-k^2\frac{\beta_1^*}{\alpha_1^*}\,\frac{\exp[-(q+q^*)h]}{|1+\beta_1/\alpha_1|^2}\right\}. \tag{53}$$

After that it is easy to obtain an expression for spectral density of energy of the electric part of fluctuating EMF. Rearranging $\tilde\alpha$ and α gives an expression for the spectral density of energy of the magnetic part. A total sum determines a spectral density of energy of the fluctuating electromagnetic field at any point over a half space and may be reduced to the form previously obtained by authors of the theory of thermal fluctuating fields [5,6]

$$U_\omega=\frac{\iota\Theta\sqrt{\epsilon\mu}}{4\pi^2c}\int_0^\infty\exp[-(q+q^*)h]\lambda\,d\lambda\left(\frac{|q|^2+\lambda^2}{k^2}+1\right)\times(S_{1\epsilon}+S_{1\mu}). \tag{54}$$

Equation (54) obviously follows any known asymptotic for propagating and evanescent fields, which may be found from the above mentioned references.

C. An application to the theory of van der Waals forces

The van der Waals force between two half spaces may be found as the z component of the Maxwell stress tensor

$$F_\omega=T_\omega^{zz}=\epsilon(\langle|E_z|^2\rangle-\langle|E_x|^2\rangle-\langle|E_y|^2\rangle)/4\pi+\mu(\langle|H_z|^2\rangle-\langle|H_x|^2\rangle-\langle|H_y|^2\rangle)/4\pi. \tag{55}$$

For different temperatures and identical materials the corresponding solution was found in [9]. Here we show that the relevant solution for different materials differs only by obvious modification. Using (27) and (30) for the case $m=0$ we may write down various forms of (55) and find the stress tensor at any appropriate surface in the transparent gap. For example, at the surface $h=0$ we have

$$\tilde F_\omega=\frac{\Theta_1}{2\pi^2\omega}\mathrm{Re}\int_0^\infty\frac{q\lambda\,d\,\lambda}{\iota}\left(\frac{\delta_2}{|D|^2}+\frac{\tilde\delta_2}{|\tilde D|^2}\right)+\frac{\Theta_2}{2\pi^2\omega}\mathrm{Re}\int_0^\infty\frac{q\lambda\,d\lambda}{\iota}\left(\frac{\delta_1}{|D|^2}+\frac{\tilde\delta_1}{|\tilde D|^2}\right) \tag{56}$$

or

$$F_\omega=-\frac{[\Theta_1+\Theta_2]}{4\pi^2\omega}\mathrm{Re}\int_0^\infty\frac{q\lambda\,d\lambda}{\iota}\left(\frac{\Delta}{D}+\frac{\tilde\Delta}{\tilde D}\right)+\frac{[\Theta_2-\Theta_1]}{4\pi^2\omega}\mathrm{Re}\int_0^\infty\frac{q\lambda\,d\lambda}{\iota}\left(\frac{\delta_1-\delta_2}{|D|^2}+\frac{\tilde\delta_1-\tilde\delta_2}{|\tilde D|^2}\right), \tag{57}$$

where

$$\Delta=\left(\frac{\beta_1}{\alpha_1}+\frac{\beta_2}{\alpha_2}\right)\sinh(ql)+\left(1+\frac{\beta_1\beta_2}{\alpha_1\alpha_2}\right)\cosh(ql),$$

$$\tilde\Delta=\left(\frac{\beta_2}{\tilde\alpha_1}+\frac{\beta_2}{\tilde\alpha_2}\right)\sinh(ql)+\left(1+\frac{\beta_1\beta_2}{\tilde\alpha_1\tilde\alpha_2}\right)\cosh(gl),$$

$$\delta_1=\frac{\beta_2}{\alpha_2}\left(\frac{q}{q^*}-\left|\frac{\beta_1}{\alpha_1}\right|^2\right),\qquad \tilde{\delta}_1=\frac{qb_2}{\tilde{\alpha}_2}\left(\frac{q}{q^*}-\left|\frac{\beta_1}{\tilde{\alpha}_1}\right|^2\right),$$

$$\delta_2=\frac{\beta_1}{\alpha_1}\left(\frac{q}{q^*}-\left|\frac{\beta_2}{\alpha_2}\right|^2\right),\qquad \tilde{\delta}_2=\frac{\beta_1}{\tilde{\alpha}_1}\left(\frac{q}{q^*}-\left|\frac{\beta_2}{\tilde{\alpha}_2}\right|^2\right). \quad (58)$$

From (56) using (B6), it is not difficult to prove that

$$\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\left(\frac{\Delta}{D}+\frac{\tilde{\Delta}}{\tilde{D}}\right)=-\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\left(\frac{\delta_1}{|D|^2}+\frac{\tilde{\delta}_1}{|\tilde{D}|^2}+\frac{\delta_2}{|D|^2}+\frac{\tilde{\delta}_2}{|\tilde{D}|^2}\right). \quad (59)$$

We represent (59) in the form $a+b=1$, where

$$a=-\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\left(\frac{\delta_1}{|D|^2}+\frac{\tilde{\delta}_1}{|\tilde{D}|^2}\right)\Big/\ \mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\times\left(\frac{\Delta}{D}+\frac{\tilde{\Delta}}{\tilde{D}}\right),$$

$$b=-\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\left(\frac{\delta_2}{|D|^2}+\frac{\tilde{\delta}_2}{|\tilde{D}|^2}\right)\Big/\ \mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\times\left(\frac{\Delta}{D}+\frac{\tilde{\Delta}}{\tilde{D}}\right). \quad (60)$$

Substituting (60) for (57) gives

$$F_\omega=-\frac{[a\Theta_1+b\Theta_2]}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota'}\left(\frac{\Delta}{D}+\frac{\tilde{\Delta}}{\tilde{D}}\right). \quad (61)$$

As follows from a comparison of (61) and (57) for identical materials, when $\delta_1=\delta_2$, $\tilde{\delta}_1=\tilde{\delta}_2$ or for the case $\Theta_1=\Theta_2$, we have $a=b=1/2$. We prove now that the same values of a and b are valid for a general case.

Let us assume for different materials that the constants a and b differ from 1/2, for example $a=1/2+a_1$ and $b=1/2+b_2$. Substitution of these values in (61) gives

$$F_\omega=-\frac{[\Theta_1+\Theta_2]}{4\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\left(\frac{\Delta}{D}+\frac{\tilde{\Delta}}{\tilde{D}}\right)-\frac{[a_1\Theta_1+b_2\Theta_2]}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\left(\frac{\Delta}{D}+\frac{\tilde{\Delta}}{\tilde{D}}\right). \quad (62)$$

After comparison of (62) with (57) with the help of (59) we have the equation

$$\frac{[a_\perp\Theta_1+b_2\Theta_2]}{2\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota}\left(\frac{\delta_1+\delta_2}{|D|^2}+\frac{\tilde{\delta}_1+\tilde{\delta}_2}{|\tilde{D}|^2}\right)=\frac{[\Theta_2-\Theta_1]}{4\pi^2\omega}\mathrm{Re}\int_0^\infty \frac{q\lambda\, d\lambda}{\iota'}\left(\frac{\delta_1-\delta_2}{|D|^2}+\frac{\tilde{\delta}_1-\tilde{\delta}_2}{|\tilde{D}|^2}\right). \quad (63)$$

Term-by-term equalizing in (63) gives the system of equations

$$\begin{aligned}\Theta_2-\Theta_1&=2(a_1\Theta_1+b_2\Theta_2),\\ \Theta_1-\Theta_2&=2(a_1\Theta_1+b_2\Theta_2).\end{aligned} \quad (64)$$

If $\Theta_1\neq\Theta_2$ the system (64) will have a solution only when $a_1=b_2=0$. It means that $a=b=1/2$ for any case and we have for different materials and temperatures the expression (61) for a spectral density of force. Finally, an integration over positive frequencies (see, for instance, [10,11]) yields the formula obtained for the case $\Theta_1=\Theta_2$ and nonmagnetic solids by Lifshitz. But for different temperatures we have

$$\begin{aligned}F=\frac{k_BT_1}{2\pi c^3}\sum_{n=0}^{\infty}\ \int_1^\infty p^2\xi_n^3\Bigg\{&\left[\left(\frac{s_{1n}+p}{s_{1n}-p}\right)\left(\frac{s_{2n}+p}{s_{2n}-p}\right)\exp(2p\xi_n l/c)-1\right]^{-1}+\left[\left(\frac{s_{1n}+\epsilon_{1n}p}{s_{1n}-\epsilon_{1n}p}\right)\left(\frac{s_{2n}+\epsilon_{2n}p}{s_{2n}-\epsilon_{2n}p}\right)\exp(2p\xi_n l/c)-1\right]^{-1}\Bigg\}dp\\
&+\frac{k_BT_2}{2\pi c^3}\sum_{m=0}^{\infty}{}'\ \int_1^\infty p^2\xi_m^3\Bigg\{\left[\left(\frac{s_{1m}+p}{s_{1m}-p}\right)\times\left(\frac{s_{2m}+p}{s_{2m}-p}\right)\exp(2p\xi_m l/c)-1\right]^{-1}+\left[\left(\frac{s_{1m}+\epsilon_{1m}p}{s_{1m}-\epsilon_{1m}p}\right)\times\left(\frac{s_{2m}+\epsilon_{2m}p}{s_{2m}-\epsilon_{2m}p}\right)\exp(2p\xi_m l/c)-1\right]^{-1}\Bigg\}dp,\end{aligned} \quad (65)$$

where $\epsilon_n=\epsilon(i\xi_n)$, $\epsilon_m=\epsilon(\iota\xi_m)$ are the values of the dielectric constants on the imaginary axis, $s_{in}=\sqrt{\epsilon_{in}-1+p^2}$, $s_{im}=\sqrt{\epsilon_{im}-1+p^2}$, $i=1,2$. The prime in the sum indicates that all terms with $n=0$ and $m=0$ have to be taken at half weight.

The formula (65) can be used to find the force for any distance, materials, and temperatures T_1 and T_2. Some asymptotic cases may be found in [9–11]. Using the solution for the plane-parallel case it is possible to obtain a related solution for solids terminated by nonplane surfaces as derived, for example, by conformal mapping [15].

D. Radiating multipole system over a half space: Energy rate liberation

In order to demonstrate an application of the obtained results concerning calculations of spatial derivatives, here we find out the energy rate liberation into a half space under the action of the origin of a harmonic electromagnetic field. We consider the case where the origin may be represented by a series of multipoles. It is clear that such a situation is relevant for scanning near-field optical microscopy (SNOM). We will obtain the result for a partial case of axial multipoles consisting of the dipole with z orientation and located in a vacuum ($\epsilon,\mu=1$) at the distance h over a half space. Taking into account our remarks at the end of Appendix C and formulas (27), (28), and (C1) for even and corresponding formulas for odd multipoles in the case for a half space ($l=\infty$), it is easy to obtain the value of energy release rate for even multipoles:

$$Q_m^e=\frac{\iota c|\mathcal{Q}_m^z|^2}{2}\int_0^\infty \lambda^3\, d\lambda\, \exp[-(q+q^*)h]|q^m|^2 S_\epsilon, \tag{66}$$

where $\mathcal{Q}_m^z$ the multipole of the m order, $m=2k$, $k=0,1,\ldots$ and for odd multipoles

$$Q_n^o=\frac{\iota c|\mathcal{Q}_n^z|^2}{2}\int_0^\infty \lambda^3\, d\lambda\, \exp[-(q+q^*)h]|q^n|^2 S_\epsilon, \tag{67}$$

where $\mathcal{Q}_n^z$ the multipole of the n order, $n=2k+1$, $k=0,1,\ldots$. It is should be clear that $\mathcal{Q}_0^z\equiv p_z$ is the dipole moment, $\mathcal{Q}_1^z$ the quadrupole moment, $\mathcal{Q}_2^z$ the octupole moment, and so on.

We consider the near-field regime and a contribution only from evanescent waves, $q+q^*=2q$, where q is the pure real number. By introducing the new variable $y=q/k_0$, $k_0=\omega/c$ we have, say, from (66)

$$Q_m^e=\frac{\iota c|\mathcal{Q}_m^z|^2 k_0^{2m+4}}{2}\int_0^\infty \exp(-2khy)(y^{2m+3}+y^{2m+1})S_e\, dy, \tag{68}$$

and a similar expression from (67).

To take the integral in (68) we simplify S_ϵ as was done in [7]. Namely, we consider two dramatically different cases, for good conductors $|\epsilon_1(\omega)|\gg 1$ and for dielectrics $\epsilon_1(\omega)\geqslant 1$, where $\epsilon_1(\omega)=\epsilon_1'(\omega)+\iota\epsilon_1''(\omega)$. For conductors and dielectrics we have, respectively,

$$\iota S_\epsilon\sim\frac{1}{y^2}\sqrt{\frac{\omega}{2\pi\sigma_1}}\quad\text{and}\quad \iota S_\epsilon\sim\frac{2\epsilon_1''(\omega)}{y|1+\epsilon_1(\omega)|^2}, \tag{69}$$

where σ_1 represents the conductivity of a half space material.

Substituting (69) for (68) and using the handbook of integrals gives us the terms of order $(2kh)^{-(2m+1)}$ for dielectrics

$$Q_m^e\simeq\frac{\omega\epsilon_1''(\omega)}{|1+\epsilon_1(\omega)|^2}\,\frac{|\mathcal{Q}_m^z|^2\Gamma(2m+3)}{2^{2m+3}h^{m+3}}, \tag{70}$$

where Γ is the gamma function, $m=2k$, $k=0,1,\ldots$.

We have the same expression for odd multipoles, but in (70) $n=2k+1$, $k=0,1,\ldots$ should be substituted.

Finally, we may write an energy release rate for even multipoles summing up all expressions like (70):

$$Q^e\simeq\frac{\omega\epsilon_1''(\omega)}{|1+\epsilon_1(\omega)|^2}\left\{\frac{|\mathcal{Q}_0^z|^2\Gamma(3)}{2^3h^3}+\frac{|\mathcal{Q}_2^z|^2\Gamma(7)}{2^7h^7}+\frac{|\mathcal{Q}_4^z|^2\Gamma(11)}{2^{11}h^{11}}+\cdots\right\}. \tag{71}$$

The corresponding expression for odd multipoles is

$$Q^o\simeq\frac{\omega\epsilon_1''(\omega)}{|1+\epsilon_1(\omega)|^2}\left\{\frac{|\mathcal{Q}_1^z|^2\Gamma(5)}{2^5h^5}+\frac{|\mathcal{Q}_3^z|^2\Gamma(9)}{2^9h^9}+\frac{|\mathcal{Q}_5^z|^2\Gamma(13)}{2^{13}h^{13}}+\cdots\right\}. \tag{72}$$

The combination of (71) and (72) gives us the general formula for the rate of energy liberation into the dielectric half space under the action of a harmonic electromagnetic field of the axial multipole system:

$$Q\simeq\frac{\omega\epsilon_1''(\omega)}{|1+\epsilon_1(\omega)|^2}\sum_{k=0}^\infty\frac{\Gamma(2k+3)}{2^{2k+3}h^{2k+3}}|\mathcal{Q}_k^z|^2. \tag{73}$$

By the same method of integration we obtain the formula for the case of good conductors:

$$Q\simeq\frac{\omega^2\zeta_1'}{c}\sum_{k=0}^\infty\frac{\Gamma(2k+2)}{2^{2k+2}h^{2k+2}}|\mathcal{Q}_k^z|^2, \tag{74}$$

where $\zeta_1'=\sqrt{\omega/8\pi\sigma_1}$, the real part of an impedance of the half space.

Obviously, it is possible to do the same calculations for multipoles composed of the p_x and p_y dipoles and find complete multipole series. This will be done in our forthcoming publications.

V. CONCLUSION

In this paper we have studied spectral properties of thermal fluctuating electromagnetic fields in a transparent plane layer between two absorbing half spaces. The materials of the half spaces are characterized by different complex electric and magnetic permeabilities. We assumed that two independent systems of external random sources of thermal fluctuating fields are distributed into the half spaces as thermostats with, in general, different constant temperatures. Spectral power densities of fluctuating electromagnetic fields and the spatial derivatives of all orders in any point of a transparent plane gap between two media were found with the help of the generalized Kirchhoff's law. In accordance with this law we calculated electromagnetic losses into the two absorbing media induced by a field of a point dipolelike or point multipolelike origins as situated in a point of interest at the transparent gap. The corresponding electrodynamical regular Green problem for a point dipole and the related problem for multipoles of any orders composed of the point dipole in the directions of interest was solved. In order to solve this problem we found an integral representation of an inhomogeneous part of a solution for point multipoles using the well-known Sommerfeld integral representation of related solution for a point dipole. We have shown various forms of obtained solutions for spectral power densities including ordinary surface reflection Fresnel coefficients for s and p polarized waves. It was demonstrated from our general solution that it is simple to obtain the known asymptotic cases. For instance, it is possible to obtain the spectral power densities of all components of fluctuating electromagnetic fields at any spatial point in a plane gap or over a half space. Using these formulas we obtained the spectral densities of energy for both propagating and evanescent waves. The Planck formula for the black body radiation directly follows from the formula for propagating waves. On the basis of the solutions obtained we found a general expression for the van der Waals forces valid for the case of different temperatures of solids. We found the rate of energy release into a half space under the action of a multipole origin of harmonic electromagnetic field and for an arbitrary origin in the case where this origin may be represented as a series of multipoles.

APPENDIX A: A PROBLEM IN REGULAR FIELDS OF MULTIPOLE ORIGINS

A common approach to solving the Maxwell equations in an inhomogeneous medium for the specified sources can be found, for example, in [12]. A solution of the boundary-value problem on the dipole field in a gap between two half spaces was obtained in [5]. We are interested in the field absorbing materials 1 and 2 (see Fig. 1); therefore, we shall seek a complete solution to this problem and extend the result to the cases of three regions with corresponding boundary conditions in the planes $z=0$ and $z=l$, and to the case of multipole origins. The solution will be found by analogy with solving the problem on a dipole above the conducting ground [12–14]; in all three media we determine the Hertz vector $\vec{Z}$. It enters by ordinary relationships with the scalar and vector potentials,

$$\varphi=-\frac{1}{\epsilon\mu}\operatorname{div}\vec{Z},\qquad \vec{A}=\frac{1}{c}\frac{\partial\vec{Z}}{\partial t}. \tag{A1}$$

Using the Lorentz condition and the expressions via the scalar and vector potentials φ and $\vec{A}$, we obtain the relation between $\vec{\mathcal{E}}$, $\vec{\mathcal{H}}$, and $\vec{Z}$. For example, in absorbing media we have

$$\vec{\mathcal{E}}^{(j)}=\frac{1}{\epsilon_j\mu_j}\{grad(\operatorname{div}\vec{Z}^{(j)})+k_j^2\vec{Z}^{(j)}\},$$

$$\vec{\mathcal{H}}^{(j)}=\frac{ik_0}{\mu_j}\operatorname{rot}\tilde{Z}^{(j)} \tag{A2}$$

where $k_0=\omega/c$ is the wave number in vacuum, $k_j^2=k_0^2\epsilon_j\mu_j$, $j=1,2$, and it is assumed that $Z\sim e^{\iota\omega t}$. By similar formulas one can find the field in the transparent gap, where $k^2=k_0^2\epsilon\mu$.

The Maxwell equations in absorbing media and in the gap can be met if the Hertz vector is known for all three media. In the case of a dipole origin, to find the Cartesian components of the Hertz vector in the gap and in the absorbing materials we need to solve the equations

$$\Delta\vec{Z}+k^2\vec{Z}=-4\pi\mu\vec{p}\,\delta(\vec{r})$$

and

$$\Delta\vec{Z}^{(j)}+k_j^2\vec{Z}^{(j)}=0, \tag{A3}$$

where $\delta(\vec{r})$ is the delta function, $\vec{r}$ the observation point, and $j=1,2$. Obviously, for the magnetic Hertz vector we need to solve the same (A3) equations replacing $\mu_i\vec{p}$ to $\epsilon_i\vec{m}$, where $\vec{m}$ is the magnetic dipole.

In the case of multipole origins we need to solve the equation

$$\Delta\vec{Z}+k^2\vec{Z}=-4\pi\mu\vec{p}(-1)^n\frac{\partial^n\delta(\vec{r})}{\partial s_1\cdots\partial s_n} \tag{A4}$$

in the gap and the corresponding homogeneous equation in absorbing half spaces.

Boundary conditions may be found as well in the classical textbook [13]. If the point dipole with the moment $p=(0,0,p_z)$ is oriented along the z axis, then the equations are satisfied given $Z=(0,0,Z_z)$. For a horizontal orientation of the dipole on the x or y axes, as shown in [13], one needs to assume, to avoid contradiction in the boundary conditions, that it is the vertical component of the Hertz vector that is induced, i.e., $\vec{Z}=(Z_x,0,\tilde{Z}_z)$ and $\vec{Z}=(0,Z_y,\tilde{Z}_z)$. Physically, this is related to the effects of media 1 and 2. In other words, one more field $\tilde{Z}$ is created by the secondary sources in media 1 and 2, which is the solution to the homogeneous equations (A3). The latter have to be completed with the boundary conditions expressing equality between the tangent components of the diffraction field at the boundaries $z=0$ and $z=l$. For the z-oriented dipole, when $\vec{Z}=(0,0,Z_z)$, we have

$$\frac{Z_z}{\mu}=\frac{Z_z^{(j)}}{\mu_j};\quad \frac{1}{\varepsilon\mu}\frac{\partial Z_x}{\partial z}=\frac{1}{\varepsilon_j\mu_j}\frac{\partial Z_z^{(j)}}{\partial z}, \tag{A5}$$

for the x-oriented dipole, when $Z=(Z_x,0,\tilde{Z}_z)$

$$Z_x=Z_x^{(j)};\quad \frac{1}{\mu}\frac{\partial Z_x}{\partial z}=\frac{1}{\mu_j}\frac{\partial Z_x^{(j)}}{\partial z};\quad \frac{\tilde{Z}_z}{\mu}=\frac{\tilde{Z}_z^{(j)}}{\mu_j}$$

$$\frac{1}{\varepsilon\mu}\left(\frac{\partial\tilde{Z}_z}{\partial z}+\frac{\partial Z_x}{\partial x}\right)=\frac{1}{\varepsilon_j\mu_j}\left(\frac{\partial\tilde{Z}_z^{(j)}}{\partial z}+\frac{\partial Z_x^{(j)}}{\partial x}\right)\quad (j=1,2). \tag{A6}$$

Identically to (A6), conditions at the boundaries are obtained for the y-oriented dipole.

APPENDIX B: EXPLICIT FORM OF A GENERAL SOLUTION

Taking into account the form of Eq. (A4), we seek the general solution of an inhomogeneous equation as a sum of a partial solution and the general solution of a homogeneous equation. Assume that in the gap between the absorbing media $Z_z = p_z Z_v$, $\tilde{Z}_z = p_x \cos\varphi \tilde{Z}_v$, $\tilde{Z}_z = p_y \sin\varphi \tilde{Z}_v$, $Z_x = p_x Z_h$, $Z_y = p_y Z_h$, where

$$Z_v = (-1)^n \mu Z_n^q + \int_0^\infty J_0(\lambda r)[G_- \exp(-qz) + G_+ \exp(qz)]\, d\lambda,$$

$$Z_h = (-1)^n \mu Z_n^q + \int_0^\infty J_0(\lambda r) O[F_- \exp(-qz) + F_+ \exp(qz)]\, d\lambda,$$

$$\tilde{Z}_v = \int_0^\infty J_1(\lambda r)[H_- \exp(-qz) + H_+ \exp(qz)]\, d\lambda, \tag{B1}$$

in absorbing materials $Z_z^{(j)} = p_z Z_v^{(j)}$, $\tilde{Z}_z^{(j)} = p_x \cos\varphi \tilde{Z}_v^{(j)}$, $\tilde{Z}_z^{(j)} = p_y \sin\varphi \tilde{Z}_v^{(j)}$, $Z_x^{(j)} = p_x Z_h^{(j)}$, $Z_y^{(j)} = p_y Z_h^{(j)}$, where

$$Z_v^{(1)} = \int_0^\infty J_0(\lambda r) G_1 \exp(q_1 z)\, d\lambda,$$

$$Z_h^{(1)} = \int_0^\infty J_0(\lambda r) F_1 \exp(q_1 z)\, d\lambda,$$

$$\tilde{Z}_v^{(1)} = \int_0^\infty J_1(\lambda r) H_1 \exp(q_1 z)\, d\lambda,$$

$$Z_v^{(2)} = \int_0^\infty J_0(\lambda r) G_2 \exp[-q_2(z-l)]\, d\lambda,$$

$$Z_h^{(2)} = \int_0^\infty J_0(\lambda r) F_2 \exp[-q_2(z-l)]\, d\lambda,$$

$$\tilde{Z}_v^{(2)} = \int_0^\infty J_1(\lambda r) H_2 \exp[-q_2(z-l)]\, d\lambda, \tag{B2}$$

where $q = \sqrt{\lambda^2 - k^2}$, $q_j = \sqrt{\lambda^2 - k_j^2}$, $(j=1,2)$. Here λ is the constant of separation in the homogeneous Helmholtz equation.

From the boundary conditions (A5) and (A6) we find the system of equations to define the coefficients $G_\pm$, $F_\pm$, $H_\pm$ G_j, F_j, H_j, $(j=1,2)$:

$$G_1 = \alpha_i[\mu\lambda(-1)^n q^{n-1} \exp(-qh) + G_+ + G_-],$$

$$G_1 = \frac{\gamma_1}{\beta_1}[\mu\lambda(-1)^n q^{n-1} \exp(-qh) + G_+ - G_-],$$

$$F_1 = \mu\lambda(-1)^n q^{n-1} \exp(-qh) + F_+ + F_-,$$

$$F_1 = \frac{\alpha_1}{\beta_1}[\mu\lambda(-1)^n q^{n-1} \exp(-qh) + F_+ - F_-],$$

$$H_1 = \alpha_1(H_+ + H_-), \tag{B3}$$

$$H_2 = \alpha_2[H_+ \exp(ql) + H_- \exp(-ql)],$$

$$G_2 = \alpha_2[\mu\lambda q^{n-1} \exp[-q(l-h)] + G_+ \exp(ql) + G_- \exp(-ql)],$$

$$G_2 = \frac{\gamma_2}{B_2}[\mu\lambda q^{n-1} \exp[-q(l-h)] + G_- \exp(-ql) - G_+ \exp(ql)],$$

$$F_2 = \mu\lambda q^{n-1} \exp[-q(l-h)] + F_+ \exp(ql) + F_- \exp(-ql),$$

$$F_2 = \frac{\alpha_2}{\beta_2}\{\mu\lambda q^{n-1} \exp[-q(l-h)] + F_- \exp(-ql) - F_+ \exp(ql)\},$$

$$\lambda F_1 - q_1 H_1 = \gamma_1\{\lambda[\mu\lambda(-1)^n q^{n-1} \exp(-qh) + F_+ + F_-] + q(H_- - H_+)\},$$

$$\lambda F_2 + q_2 H_2 = \gamma_2\{\lambda[\mu\lambda q^{n-1} \exp(-q(l-h)) + F_+ \exp(ql) + F_- \exp(-ql)] + q[H_- \exp(-ql) - H_+ \exp(ql)]\},$$

where

$$\beta_j = \frac{q_j}{q}, \quad \gamma_j = \frac{k_j^2}{k^2} = \frac{\varepsilon_j \mu_j}{\varepsilon\mu} = \alpha_j \tilde{\alpha}_j, \quad \alpha_j = \frac{\mu_j}{\mu},$$

$$\tilde{\alpha}_j = \frac{\varepsilon_j}{\varepsilon} \quad (j=1,2). \tag{B4}$$

Thus, we have 12 equations in this system with 12 unknown coefficients.

For solving of this system it is convenient to separate even $m=2k$ and odd $n=2k+1$ $(k=0,1,...)$ parts of a solution. Finally, we have

$$G_1^e = \frac{2\mu_1\lambda q^{m-1}}{\tilde{D}}\left[\cosh(ql-qh) + \frac{\beta_2}{\tilde{\alpha}_2}\sinh(ql-qh)\right],$$

$$G_1^0 = -\frac{2\mu_1\lambda \dot{q}^{n-1}}{\tilde{D}}\left[\sinh(ql-qh) + \frac{\beta_2}{\tilde{\alpha}_2}\cosh(ql-qh)\right],$$

$$G_2^e = \frac{2\mu_2\lambda q^{m-1}}{\tilde{D}}\left[\cosh(qh) + \frac{\beta_1}{\tilde{\alpha}_1}\sinh(qh)\right],$$

$$G_2^o = \frac{2\mu_2^\lambda \lambda q^{n-1}}{\tilde{D}}\left[\sinh(qh) + \frac{\beta_1}{\tilde{\alpha}_1}\cosh(qh)\right],$$

$$F_1^e=\frac{2\mu\lambda q^{m-1}}{D}\left[\cosh(ql-qh)+\frac{\beta_2}{\alpha_2}\sinh(ql-qh)\right], \tag{B5}$$

$$F_1^o=-\frac{2\mu\lambda q^{n-1}}{D}\left[\sinh(ql-qh)+\frac{\beta_2}{\alpha_2}\cosh(ql-qh)\right],$$

$$F_2^e=\frac{2\mu\lambda g^{m-1}}{D}\left[\cosh(qh)+\frac{\beta_1}{\alpha_1}\sinh(qh)\right],$$

$$F_2^o=\frac{2\mu\lambda q^{n-1}}{D}\left[\sinh(qh)+\frac{\beta_1}{\alpha_1}\cosh(qh)\right],$$

$$H_1^{eo}=\frac{\alpha_1\lambda}{q\tilde{D}}\left\{F_1^{eo}\left(\frac{1-\gamma_1}{\gamma_1}\right)\left[\cosh(ql)+\frac{\beta_2}{\tilde{\alpha}_2}\sinh(ql)\right]\right.$$
$$\left.-F_2^{eo}\left(1-\frac{\gamma_2}{\gamma_2}\right)\right\},$$

$$H_2^{eo}=\frac{\alpha_2\lambda}{q\tilde{D}}\left\{F_1^{eo}\left(\frac{1-\gamma_1}{\gamma_1}\right)-F_2^{eo}\left(\frac{1-\gamma_2}{\gamma_2}\right)\left[\cosh(ql)\right.\right.$$
$$\left.\left.+\frac{\beta_1}{\tilde{\alpha}_1}\sinh(ql)\right]\right\},$$

where

$$D=\left(\frac{\beta_1}{\alpha_1}+\frac{\beta_2}{\alpha_2}\right)\cosh(ql)+\left(1+\frac{\beta_1\beta_2}{\alpha_1\alpha_2}\right)\sinh(ql),$$

$$\tilde{D}=\left(\frac{\beta_1}{\tilde{\alpha}_1}+\frac{\beta_2}{\tilde{\alpha}_2}\right)\cosh(ql)+\left(1+\frac{\beta_1\beta_2}{\tilde{\alpha}_1\tilde{\alpha}_2}\right)\sinh(ql). \tag{B6}$$

APPENDIX C: CALCULATION OF ELECTROMAGNETIC LOSSES

Using formulas for components of a field (A2) in the cylindrical system of coordinates and obtained expressions for components of the Hertz vector we find losses in both half spaces of different multipoles located in a plain gap in accordance with (16). For even and odd multipoles composed of the p_z component of a unit dipole we have

$$Q_{1z}^{ee}=-\mathrm{Re}\frac{\iota\omega|p_z|^2}{4\epsilon_1|\mu_1|^2}\int_0^\infty\lambda\,d\lambda[q_1|G_1^e|^2], \tag{C1}$$

$$Q_{2z}^{ee}=-\mathrm{Re}\frac{\iota\omega|p_z|^2}{4\epsilon_2|\mu_2|^2}\int_0^\infty\lambda\,d\lambda[q_2|G_2^e|^2]. \tag{C2}$$

For even multipoles composed of $p_{x,y}$ components,

$$Q_{1xy}^{ee}=-\mathrm{Re}\frac{\iota\omega|p_{x,y}|^2}{8\epsilon_1|\mu_1|^2}\int_0^\infty\lambda\,d\lambda\left[|F_1^e|^2q_1^*\frac{(q_1^2-k_1^2)}{\lambda^2}+|H_1^e|^2q_1\right.$$
$$\left.-F_1^{e*}H_1^e\frac{|q_1|^2}{\lambda}-F_1^eH_1^{e*}\frac{q_1^2}{\lambda}\right], \tag{C3}$$

$$Q_{2xy}^{ee}=-\mathrm{Re}\frac{\iota\omega|p_{x,y}|^2}{8\epsilon_2|\mu_2|^2}\int_0^\infty\lambda\,d\lambda\left[|F_2^e|^2q_2^*\frac{(q_2^2-k_2^2)}{\lambda^2}+|H_2^e|^2q_2\right.$$
$$\left.+F_2^{e*}H_2^e\frac{|q_2|^2}{\lambda}+F_2^eH_2^{e*}\frac{q_2^2}{\lambda}\right]. \tag{C4}$$

For odd multipoles we have the above formulas, but with corresponding odd coefficients $G_{1,2}^o$, $F_{1,2}^o$, and $H_{1,2}^o$.

For mixed even-odd (e^*o) multipoles composed of p_z component we have

$$Q_{1z}^{e^*o}=-\mathrm{Re}\frac{\iota\omega|p_z|^2}{4\epsilon_1|\mu_1|^2}\int_0^\infty\lambda\,d\lambda[q_1G_1^{e*}G_1^o], \tag{C5}$$

$$Q_{2z}^{e^*o}=-\mathrm{Re}\frac{\iota\omega|p_z|^2}{4\epsilon_2|\mu_2|^2}\cdot\int_0^\infty\lambda\,d\lambda[q_2G_2^{e*}G_2^o]. \tag{C6}$$

For mixed even–odd (e^*o) multipoles constituted by $p_{x,y}$ components we obtain

$$Q_{1xy}^{e^*o}=-\mathrm{Re}\frac{\iota\omega|p_{x,y}|^2}{8\epsilon_1|\mu_1|^2}\int_0^\infty\lambda\,d\lambda\left[F_1^{e*}F_1^oq_1^*\frac{(q_1^2-k_1^2)}{\lambda^2}\right.$$
$$\left.+H_1^{e*}H_1^oq_1-F_1^{e*}H_1^o\frac{|q_1|^2}{\lambda}-H_1^{e*}F_1^o\frac{q_1^2}{\lambda}\right], \tag{C7}$$

$$Q_{2xy}^{e^*o}=-\mathrm{Re}\frac{\iota\omega|p_{x,y}|^2}{8\epsilon_2|\mu_2|^2}\int_0^\infty\lambda\,d\lambda\left[F_2^{e*}F_2^oq_2^*\frac{(q_2^2-k_2^2)}{\lambda^2}\right.$$
$$\left.+H_2^{e*}H_2^oq_2+F_2^{e*}H_2^o\frac{|q_2|^2}{\lambda}+H_2^{e*}F_2^o\frac{q_2^2}{\lambda}\right]. \tag{C8}$$

For mixed even–odd (eo^*) multipoles we have the same formulas with complex conjugated coefficients $G_{1,2}^eG_{1,2}^{o*}$, $F_{1,2}^eF_{1,2}^{o*}$, $H_{1,2}^eH_{1,2}^{o*}$, and $F_{1,2}^eH_{1,2}^{o*}$.

Thus, we found the electromagnetic losses in two half spaces originating from point multipoles of any order. As an important limiting case it is clear to obtain corresponding losses of a multipole situated over a half space.

It should be emphasized that we found the losses that may be used for different practical cases. When $\vec{p}=\vec{l}/\iota\omega$ is the unit dipole, where $\vec{l}$ the unit dimensionless vector, the dimensionality of this dipole is $[p]=\mathrm{s\,rad}^{-1}$ and the dimensionality of the losses corresponding to this dipole is $[Q]=\mathrm{s\,cm}^{-3}$ in accordance with the requirement of the generalized Kirchhoff's law, see [6], or the Eq. (11). However, if we use an ''ordinary'' dipole with the dimensionality $[p]$

$=g^{1/2}\,\mathrm{cm}^{5/2}\,\mathrm{s}^{-1}$ we have ordinary losses with the dimensionality $[Q]=\mathrm{erg\,s}^{-1}$. The same, obviously, is applicable to multipole losses. As follows from (21), in order to obtain the ordinary losses $[Q]=\mathrm{erg\,s}^{-1}$, say, from the quadrupole composed of the p_z dipole

$$\mathcal{Q}_1^z=\lim_{|\Delta z|\to 0}(p_z\Delta z), \tag{C9}$$

we must simply substitute the value of this quadrupole into (C1), (C2) instead of p_z etc., for any multipole.

From the above it follows the way to obtain relevant formulas for the important case, when a multipole is situated at some distance over a half space.

APPENDIX D: CALCULATED COEFFICIENTS FOR SPECTRAL CHARACTERISTICS

The calculation discussed in this paper gave the following coefficients for the even derivatives of the x,y components of the electric field:

$$C_m^e=|q^m|^2\frac{\left|\cosh(ql-qh)+\dfrac{\beta_2}{\alpha_2}\sinh(ql-qh)\right|^2}{|D|^2},$$
$$\tilde{C}_m^e=|q^m|^2\frac{\left|\sinh(ql-qh)+\dfrac{\beta_2}{\tilde{\alpha}_2}\cosh(ql-qh)\right|^2}{|\tilde{D}|^2}, \tag{D1}$$

$$D_m^e=|q^m|^2\frac{\left|\cosh(qh)+\dfrac{\beta_1}{\alpha_1}\sinh(qh)\right|^2}{|D|^2},$$
$$\tilde{D}_m^e=|q^m|^2\frac{\left|\sinh(qh)+\dfrac{\beta_1}{\tilde{\alpha}_1}\cosh(qh)\right|^2}{|\tilde{D}|^2},\quad m=2k,$$
$$k=0,1,\ldots .$$

For the odd derivatives of the x,y components we have coefficients

$$C_n^o=|q^n|^2\frac{\left|\sinh(ql-qh)+\dfrac{\beta_2}{\alpha_2}\cosh(ql-qh)\right|^2}{|D|^2},$$
$$\tilde{C}_n^o=|q^n|^2\frac{\left|\cosh(ql-qh)+\dfrac{\beta_2}{\tilde{\alpha}_2}\sinh(ql-qh)\right|^2}{|\tilde{D}|^2},$$
$$D_n^o=|q^n|^2\frac{\left|\sinh(qh)+\dfrac{\beta_1}{\alpha_1}\cosh(qh)\right|^2}{|D|^2}, \tag{D2}$$
$$\tilde{D}_n^o=|q^n|^2\frac{\left|\cosh(qh)+\dfrac{\beta_1}{\tilde{\alpha}_1}\sinh(qh)\right|^2}{|\tilde{D}|^2},\quad n=2k+1,\quad k=0,1,\ldots .$$

For mixed odd-even derivatives $(o*e)$ of the z components of a field we obtain

$$A_{mn}^{o*e}=-q^m q^{*n}\frac{\left[\cosh(ql-qh)+\dfrac{\beta_2}{\tilde{\alpha}_2}\sinh(ql-qh)\right]\left[\sinh(q^*l-q^*h)+\dfrac{\beta_2^*}{\tilde{\alpha}_2^*}\cosh(q^*l-q^*h)\right]}{|\tilde{D}|^2}$$
$$B_{mn}^{o*e}=q^m q^{*n}\frac{\left[\cosh(qh)+\dfrac{\beta_1}{\tilde{\alpha}_1}\sinh(qh)\right]\left[\sinh(q^*h)+\dfrac{\beta_1^*}{\tilde{\alpha}_1^*}\cosh(q^*h)\right]}{|\tilde{D}|^2}. \tag{D3}$$

For mixed odd-even derivatives $(o*e)$ of the x,y components of a field we obtain

$$C_{mn}^{o*e}=q^m q^{*n}\frac{\left[\cosh(ql-qh)+\frac{\beta_2}{\alpha_2}\sinh(ql-qh)\right]\left[\sinh(q^*l-q^*h)+\frac{\beta_2^*}{\alpha_2^*}\cosh(q^*l-q^*h)\right]}{|D|^2},$$

$$\tilde{C}_{mn}^{o*e}=q^m q^{*n}\frac{\left[\sinh(ql-qh)+\frac{\beta_2}{\tilde{\alpha}_2}\cosh(ql-qh)\right]\left[\cosh(q^*l-q^*h)+\frac{\beta_2^*}{\tilde{\alpha}_2^*}\sinh(q^*l-q^*h)\right]}{|\tilde{D}|^2}.$$

$$D_{mn}^{o*e}=q^m q^{*n}\frac{\left[\cosh(qh)+\frac{\beta_1}{\alpha_1}\sinh(qh)\right]\left[\sinh(q^*h)+\frac{\beta_1^*}{\alpha_1^*}\cosh(q^*h)\right]}{|D|^2},$$

$$\tilde{D}_{mn}^{o*e}=q^m q^{*n}\frac{\left[\sinh(qh)+\frac{\beta_1}{\tilde{\alpha}_1}\cosh(qh)\right]\left[\cosh(q^*h)+\frac{\beta_1^*}{\tilde{\alpha}_1^*}\sinh(q^*h)\right]}{|\tilde{D}|^2}. \tag{D4}$$

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